Continuous Models 1: ODE

- in scientific computing: *numerical* simulations and, hence, typically *continuous* models
- two big classes:
  - problems *with* a treatment of space (involving *partial* differential equations (PDE))
  - problems *without* a treatment of space (involving *ordinary* differential equations (ODE))
- standard example for the latter: population dynamics
  - development (growth) of populations,
    - either isolated (without external influences)
    - or in coexistence (peaceful or hostile) of different species
  - modelling has a long tradition
  - classical representative: model of Maltus (1798)

Model of Maltus

- one species considered:
  - constant birth rate γ per time unit and individual
  - constant death rate δ per time unit and individual
  - thus, constant growth rate $\lambda = \gamma - \delta$
- development of p(t), the number of individuals:

$$p(t + \Delta t) = p(t) + \lambda p(t) \Delta t$$

(growth is proportional to size of population and time)

- this leads to the ODE $\dot{p}(t) = \lambda\, p(t)$ with solution $p(t) = p_0 e^{\lambda t}$ if $p(0) = p_0$
- note:
  - exponential growth or decrease
  - *discrete* reality, but *continuous* model!

Model Refinement 1

- Is exponential growth realistic?
  - population of the earth between 1700 and 1960: yes!
    - growth rate of about 0.02
    - population doubles in 34.67 years
  - generally: no!
    - limited capacity of the earth, limited resources
    - increasing competiton for food, water, or air slows down growth
- refinement following Verhulst and others (19. cent.):
  - population tends towards some saturation limit
  - *linear* birth and death rates (now per time unit only):

  $$\gamma(t) = \gamma_0 - \gamma_1 p(t) \qquad \delta(t) = \delta_0 + \delta_1 p(t) \qquad \gamma_0 > \delta_0 > 0, \gamma_1, \delta_1 > 0$$

    big population decreases birth rate and increases death rate
  - Limit exists if time tends towards infinity!

Model Refinement 2

- Is Verhulst‘s approach realistic?
  - Second derivative of p(t) does not change its sign!
  - S-shape is widespread, however (US-population 1790-1950)
- further refinement in order to obtain S-shape:

  $$\dot{p}(t) = a \cdot p(t) - b \cdot p^2(t), \quad a > b > 0$$

  with solution $$p(t) = \frac{a \cdot p_0}{b \cdot p_0 + (a - b \cdot p_0) e^{-at}}$$
- discussion:
  - limit for t towards infinity is a/b
  - S-shape for $p_0 < a / b$
  - example USA 1790-1950: a=0.03134, b=0.000000000156
  - better than our starting point, but still no external influences

Model Refinement 3

- Typically, a is much bigger than b:
  - The quadratic term gets influence only for really big p(t).
  - Why quadratic and not cubic? – That's modelling!
  - justification: an individual is disturbed proportional to p(t)
- S-shape growth is called logistic
- For that, other ODEs can be used, too (widespread in modelling of tumours' growth, e.g.):
  - $\dot{p}(t) = \lambda(t) \cdot p(t)$ with some continuous, positive, and decreasing function lambda (empirically, growth rates have to decrease)
  - or, even more general, $\dot{p}(t) = f(p(t)) \cdot p(t)$ with some suitable non-negative, decreasing, and vanishing (for increasing t) f

More than one Species

- next step of refinement: consider two species p and q

  $$\dot{p}(t) = f(p(t), q(t)) \cdot p(t),$$
  $$\dot{q}(t) = g(p(t), q(t)) \cdot q(t)$$

  f and g defined for positive p and q
- A solution is a limit or stationary situation (no growth)

  $$f(\bar{p}, \bar{q}) = g(\bar{p}, \bar{q}) = 0, \quad \bar{p}, \bar{q} > 0$$

  and is called *equilibrium* (stable state of the system).
- Is there an equilibrium? If yes, is it *attractive* ?
  - theory of ODEs: a sufficient condition are negative real parts of the eigenvalues of the Jacobian of $F(p,q) = (f(p,q)p, g(p,q)q)$
  - We study two special cases: *predator-prey* and *competition* characteristics.

Competition Characteristics

- The species p and q do not „eat" each other, but both struggle for the same resources:
  $$f_p(p,q), f_q(p,q), g_p(p,q), g_q(p,g) < 0 \text{ for } p,q > 0$$
- Sufficient condition for attractive equilibrium reads
  $$f_p(\overline{p},\overline{q}) \cdot g_q(\overline{p},\overline{q}) - f_q(\overline{p},\overline{q}) \cdot g_p(\overline{p},\overline{q}) > 0$$
  (p's influence on p is bigger than on q and vice versa)
- simple concrete choice for f and g:
  $$f(p,q) = a_1 + a_2 \cdot p + a_3 \cdot q, \quad g(p,q) = a_4 + a_5 \cdot p + a_6 \cdot q$$
  with (due to our model assumptions)
  $$a_1, a_4 > 0, \quad a_2, a_3, a_5, a_6 < 0, \quad a_2 \cdot a_6 > a_3 \cdot a_5$$
- Attractive equil. $f(\overline{p},\overline{q}) = g(\overline{p},\overline{q}) = 0, \quad \overline{p},\overline{q} > 0$ exists!

Predator-Prey Characteristics

- q is „food" of p, which leads to a different growth behaviour:
  $$f_p(p,q), g_p(p,q), g_q(p,q) < 0, \ f_q(p,q) > 0 \text{ for } p,q > 0$$
  $$a_2, a_5, a_6 < 0, \quad a_3 > 0$$
  (p, of course, enjoys an increasing population q)
- Sufficient condition for attractive equilibrium is always fulfilled (i.e., if there is an equilibrium, it is attractive).
- classical representative: model of Volterra and Lotka:
  $$a_2 = a_6 = 0$$
  - no influence of p on p or q on q
  - There is an attractive equilibrium (though our sufficient condition is not valid)!